Math Senior Project:
Golf Scheduling Problems

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Contents

1 Introduction

Consider a golf tournament in which twelve players split up into three groups of four to play five rounds. (Groups can be different in each round.) Is it possible for each player to play in a group with each other player at least once, but no more than twice? We explore and expand this problem, presenting computer algorithms and mathematical results.

In general, a golf schedule has x people per group, y groups, z rounds, and r times each player plays with each other one. In the motivating problem discussed above we have $x = 4$, $y = 3$, $z = 5$, and $1 \leq r \leq 2$.

2 Computer search

In this section we discuss how a computer search can be used to find golf schedule. We will use the motivating problem as an example, but many of the same techniques could be used for other given constraints.

Since we are using a computer, let's find a way to represent rounds in a computer. We can assign each of the xy players an element in a set. The elements can be arranged in a list, assigning the first x letters to group 1, the second x to group 2, and so forth. This is not unique since

$$\begin{array}{lll} ABCD & |\,EFGH & |\,IJKL \\ DCBA & |\,HGFE & |\,LKJI \\ EFGH & |\,IJKL & |\,ABCD \end{array}$$

are all representations of the same round. We need to have a way to count the number of different rounds.

Theorem 2.1. If there are y groups of x people, then there are $\frac{(xy)!}{(x!)^y y!}$ different rounds.

Proof. We can represent players in a round as a list described above. There are $(xy)!$ different ways to permute this list. But we are counting repeats. There are $x!$ ways to order the x players in each of the y groups. Also, there are $y!$ different ways to order the y groups. Therefore, in the general case, there are $\frac{(xy)!}{(x!)^y y!}$ different rounds. □

In particular, the motivating problem has $\frac{12!}{(4!)^3 3!}$ different rounds. Checking would be $5775^5 = 6.42 \times 10^{18}$ computations, which is a very large number. On my computer it takes approximately 8.31×10^{-6} seconds per computation. This gives us a program with a runtime of around 1.70 million years. That is probably going to be too long.

If two groups are the same in two different rounds, then in the other rounds the four players must be split up among three groups. By the pigeonhole principal, a third occurrance of a pair is force. Since no groups can be the same in different rounds, not two rounds may be the same. Therefore, repeats of rounds can be eliminated. Also, since we just want 5 rounds, the order of the rounds does not matter. With these two improvements we only need $\binom{5775}{5} = 5.34 \times 10^{16}$ calculations, but this only reduces the runtime to about 14,100 years, which is still extremely long.

We can reduce the search space even further by choosing an arbitrary first round, $ABCD \mid EFGH \mid IJKL$, since before the first round nothing distiguishes any of the players. So we only need to choose 4 from 5774 different rounds. This gives us $\binom{5774}{4} = 4.63 \times 10^{13}$ computations, and a search time of 12.2 years. Since this is only year long senior project, this is still too long.

We are able to reduce the search time to a time frame that is just over a decade, but we still don't want to have to wait that long to get a single result. When we choose an arbitrary first round, we are selecting a representative from the only symmetry class of the first round. We can apply the same concept on the second round, by dividing all the possible second rounds into symmetry classes. A representative from each of the symmetry classes is shown in Figure 1. We do not have to choose a representative from the first symmetry class since it is a repeat of the first round. So this reduction gives us $8 \cdot \binom{5773}{3} = 2.56 \times 10^{11}$ computations, and a search time of 24.6 days, which is just under a month. This is much more reasonable, but we can still do better.

First round: $ABCD \mid EFGH \mid IJKL$

$ABCD$	$EFGH$	$IJKL$
$ABCD$	$EFGI$	$HJKL$
$ABCD$	$EFIJ$	$GHKL$
$ABCE$	$DFGI$	$HJKL$
$ABCE$	$DFIJ$	$GHKL$
$ABCE$	$DIJK$	$FGHL$
$ABEF$	$CDIJ$	$GHKL$
$ABEF$	$CGIJ$	$DHKL$
$ABEI$	$CFGJ$	$DHKL$

Figure 1: Second round symmetry representatives

Checking after choosing three or four rounds, if either of the constraints have been broken can improve the search time by short-circuiting. If more pairings are required than possible for everyone to play everyone else at least once, then we do not have to check the remaining rounds. If one pairing has occured three times, then we also do not have to check the remaining rounds. This obviously improves the search time, but an exact improvement can not be calculated because we do not know which rounds we can skip.

Running a program wich all of these improvements takes a little over 2 hours on my computer. Refer to Figure 2 to compare the different improvements to the search algorithm.

Improvement	Number of computations	Time
None	$5775^5 = 6.42 \times 10^{18}$	1,700,000 years
No repetition or order	$\binom{5775}{5} = 5.34 \times 10^{16}$	14,100 years
First round arbitrary	$\binom{5774}{4} = 4.63 \times 10^{13}$	12.2 years
Second round symmetry	$8 \cdot \binom{5773}{3} = 2.56 \times 10^{11}$	24.6 days
Check if still possible	???†	2.12 hours†

†Observed calculations

Figure 2: Search time comparison

An exhaustive search concludes that a schedule with these constraints does not exist. Knowing the solution to the motivating problem, let's find a combinatorial argument to further confirm our computer result and add some elegance.

In each group there are $\binom{4}{2}$ pairings and there are 3 groups per round and 5 rounds. This gives us $\binom{4}{2} \cdot 3 \cdot 5 = 90$ pairings in the entire schedule. For everyone to play everyone exactly once we require $\binom{12}{2} = 66$ pairings. Therefore out of the 90 pairings, $90 - 66 = 24$ of the pairings must be the second occurrence of a pairing.

If we split a group of four from one round into the three groups in the following rounds, by the pigeonhole principle, at least one of the pairings will repeat in each round. Therefore each of the 3 groups in the first round will have a minimum of 4 forced second occurrences of a pairing in the remaining 4 rounds. Similarly in each of the following rounds each group will have a pairing forced in the rounds after it. So we have at least $3(4 + 3 + 2 + 1) = 30$ second occurrences of pairings. Since $30 > 24$ we have a contradiction and a schedule with $x = 4$, $y = 3$, $z = 5$, and $1 \leq r \leq 2$ does not exist.

We can revise the computer search to find the next best solution by allowing everyone play everyone else at least once, but no more than three times. This does allow a schedule. One such schedule is shown in Figure 3.

Schedule:

ABCD	*EFGH*	*IJKL*
ABEI	*CFGJ*	*DHKL*
ABFK	*CEHL*	*DGIJ*
AEHJ	*BGKL*	*CDFI*
AFGL	*BHIJ*	*CDEK*

Black: one pairing
Red: two pairings
Blue: Three pairings

Relations												
A:		*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
B:	*A*		*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
C:	*A*	*B*		*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
D:	*A*	*B*	*C*		*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
E:	*A*	*B*	*C*	*D*		*F*	*G*	*H*	*I*	*J*	*K*	*L*
F:	*A*	*B*	*C*	*D*	*E*		*G*	*H*	*I*	*J*	*K*	*L*
G:	*A*	*B*	*C*	*D*	*E*	*F*		*H*	*I*	*J*	*K*	*L*
H:	*A*	*B*	*C*	*D*	*E*	*F*	*G*		*I*	*J*	*K*	*L*
I:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*		*J*	*K*	*L*
J:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*		*K*	*L*
K:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*		*L*
L:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	

Figure 3: Representitive schedule

3 Schedules with $r = 1$

In this section we consider various values of x and y, restricting r to 1. Let us then try to characterize the values of x and y that yield a schedule.

Clearly, if there is one group, everyone plays with everyone else exactly once in one round. Also, if we have one person per group and two or more groups, it is impossible for anyone to play with anyone else. We fill these values in the first version of the grid in Figure 4, but the rest is still unknown. We continue to revisit this grid as our knowledge expands.

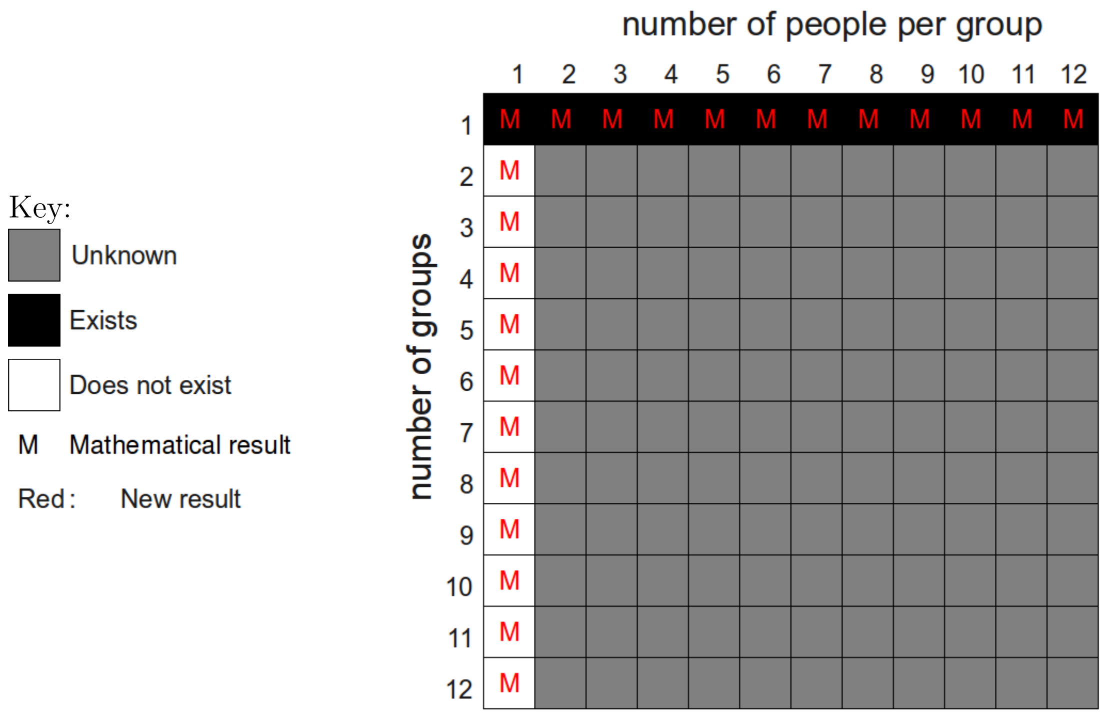


Figure 4: Schedules with $r = 1$, version 1

The program that solved the motivating problem and its relaxation can be used to search for schedules with $r = 1$. Again we consider how many computations this requires. Figure 5 shows that even with modest values of x and y, we have a huge search space.

Number of groups	Number of people per group: 1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	3	10	35	126	462	1716
3	1	1365	3.6e6	1.7e7	8.0e9	4.1e12	2.2e15
4	1	1.6e9	2.3e15	2.0e24	1.9e25	1.5e32	1.3e39
5	1	1.5e19	1.1e34	8.9e44	3.2e61	7.1e62	2.0e76
6	1	4.0e33	1.8e54	1.2e73	4.9e99	3.6e126	2.0e128
7	1	7.7e52	3.0e89	2.3e124	4.0e148	2.5e188	2.8e288

Figure 5: Combinatorial explosion

Results show that a search space of magnitude 10^{35} is the largest is reasonable (taking less than a week). This is much larger than in the previous section, but before we had a range for r. In this section, the search is improved because we can short-circuit once amu pair has occured a second time. Our grid, updated with the results obtained by computer, is in Figure 6.

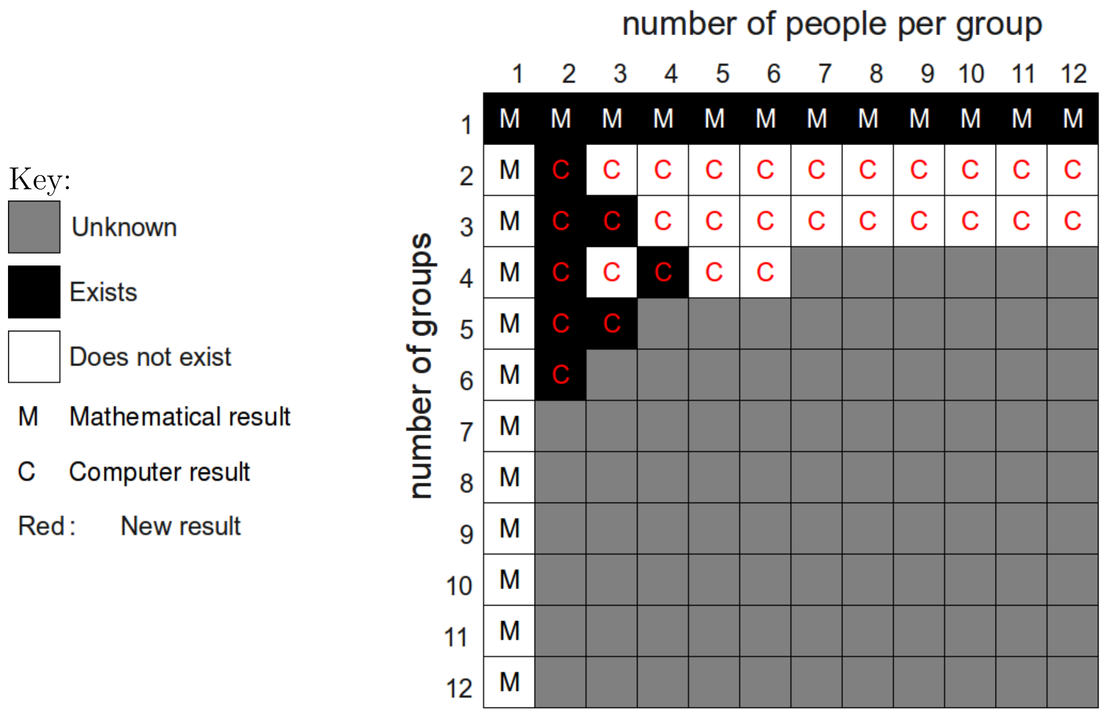


Figure 6: Schedules with $r = 1$, version 2

At this point we shift our attention from a computer search to explore some mathematical results. Since everyone plays everyone else exactly once, we can calculate how many rounds are played. Since we can only have an integer number of rounds, if a fraction of a round is required, we know that schedule does not exist.

Theorem 3.1. For x people per group, y groups, and an arbitrary r, there are $z = r\frac{xy-1}{x-1}$ rounds.

Proof. A player needs to play with $xy - 1$ other players r times. In each round they will play with $x - 1$ other players. So if a schedule exists we need $z = r\frac{xy-1}{x-1}$ rounds. □

If r is a range we allow the number of rounds to be a range, rounding up for the lower bound and down for the upper bound. For the original motivating problem we have $4 \leq r \leq 7$. If $r = 1$, then $z = \frac{xy-1}{x-1}$. This number needs to be an integer. In Figure 7 we rule out all the combinations of x and y where $\frac{xy-1}{x-1}$ is not an integer.

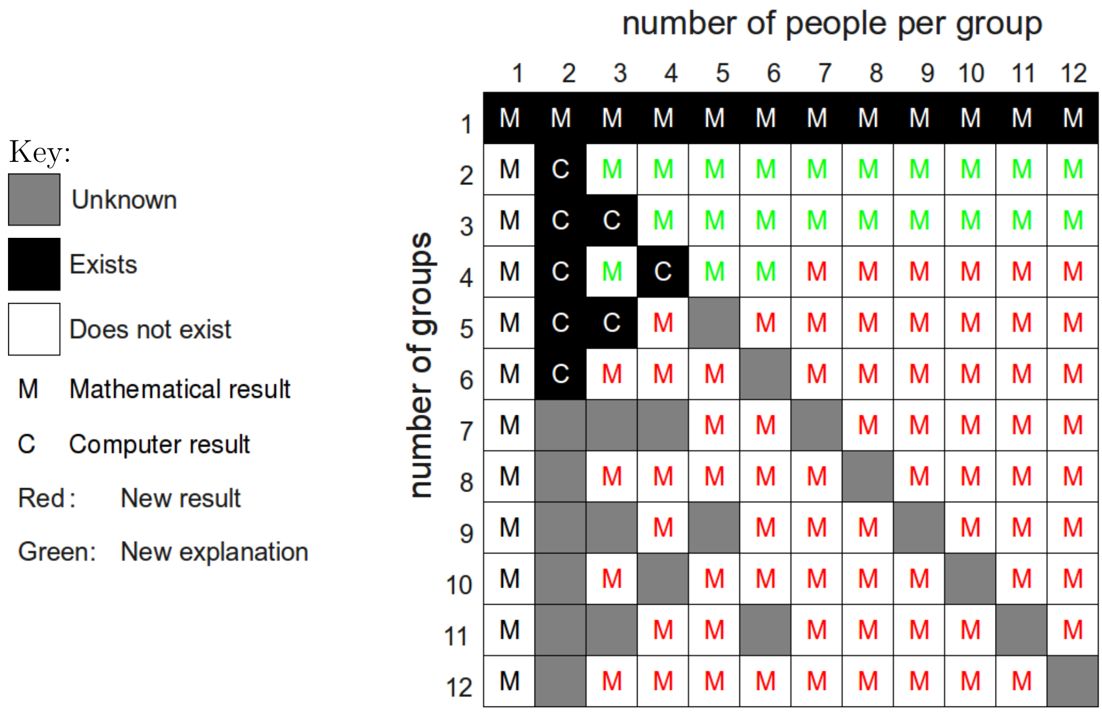


Figure 7: Schedules with $r = 1$, version 3

Referring to Figure 7, we see that many of the sechedules fall into one of two classes. The first is the vertical line with $x = 2$. The second is along the diagonal with $x = y$. We show that many sechedules in these two classes exist.

Theorem 3.2. For $x = 2$ and $r = 1$, there exists a schedule.

Proof. We construct a schedule by representing is as an edge-colored complete graph. Place one vertex in the center and the remainder equally spaced around the "rim". Color each "spoke" and the edges perpendicular to it the same color. Let the vertices correspond to players, edges correspond to groups, and colors correspond to rounds.

Each group will have two players, since each edge consists of exactly two vertices. The center vertex obviously has each edge a different color. An outer vertex could not be on two edges of the same color because given a point an a line, there exist exactly one perpendicular line from that point through that line. Since it is a complete graph any two players will play together. Therefore the constraints are satisfied. □

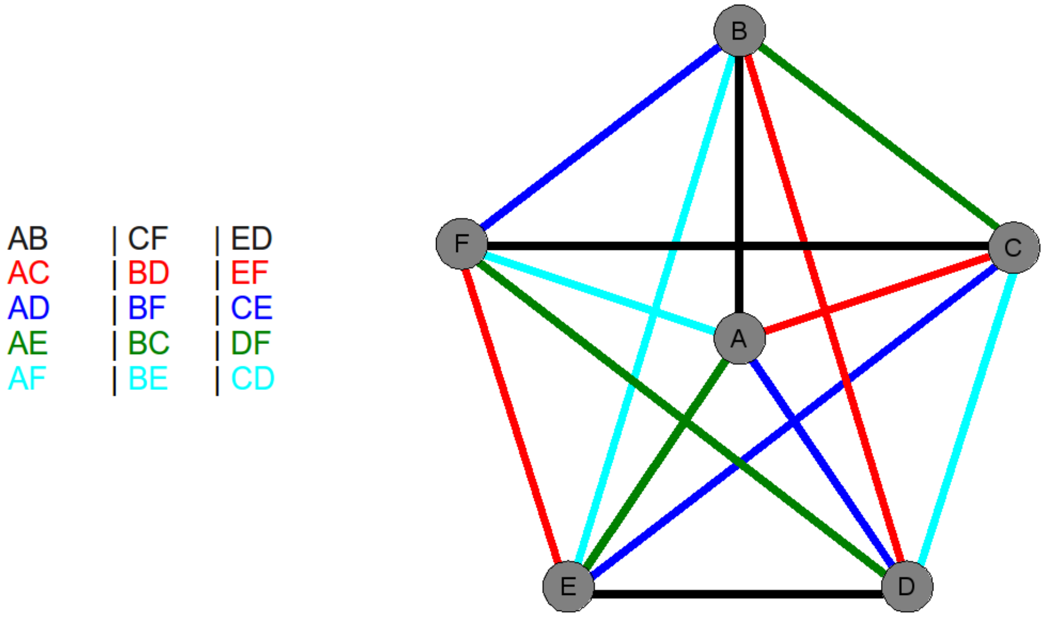
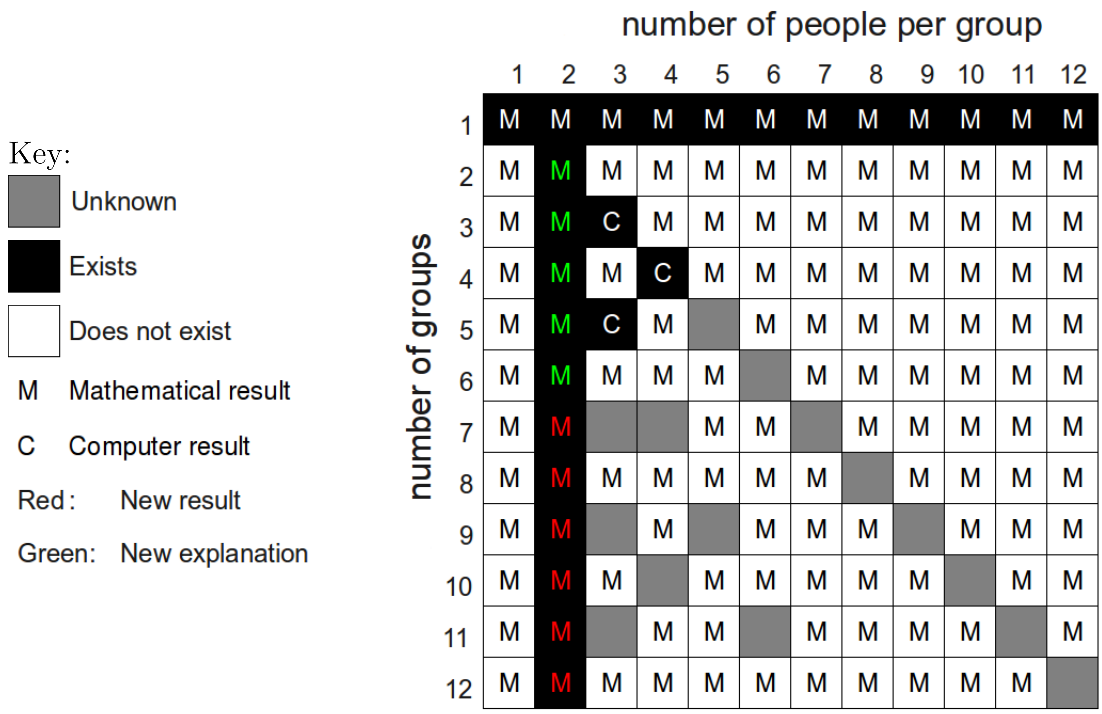


Figure 8: Example of an edge-colored graph of an $x = 2$ and $y = 3$ schedule



	1	2	3	4	5	6	7	8	9	10	11	12
1	M	M	M	M	M	M	M	M	M	M	M	M
2	M	M	M	M	M	M	M	M	M	M	M	M
3	M	M	C	M	M	M	M	M	M	M	M	M
4	M	M	M	C	M	M	M	M	M	M	M	M
5	M	M	C	M		M	M	M	M	M	M	M
6	M	M	M	M	M		M	M	M	M	M	M
7	M	M			M	M		M	M	M	M	M
8	M	M	M	M	M	M	M		M	M	M	M
9	M	M		M		M	M	M		M	M	M
10	M	M	M		M	M	M	M	M		M	M
11	M	M		M	M		M	M	M	M		M
12	M	M	M	M	M	M	M	M	M	M	M	

Figure 9: Schedules with $r = 1$, version 4

We can shift our focus to the second class of schedules, the diagonal where $x = y$.

Theorem 3.3. Let $x = y = p^n$ such that p is prime and n is a positive integer. Then there exists a schedule with $r = 1$.

Proof. Let players correspond to elements of $GF(p^n) \times GF(p^n)$,[1] rounds correspond to elements of $GF(p^n) \cup \{*\}$, and groups correspond to elements of $GF(p^n)$. In group γ of round ρ, let the set of players is be given

$$\phi(\rho,\gamma) = \begin{cases} \{(\gamma,i) | i \in GF(p^n)\} & : \rho = * \\ \{(i,\gamma + i \cdot \rho) | i \in GF(p^n)\} & : \rho \neq * \end{cases}.$$

Given a group γ and round ρ clearly p^n players are in the group. Consider a round ρ and a player (s,t). If $\rho = *$, then (s,t) plays in group s. If $\rho \neq *$, then (s,t) plays in group $t - s \cdot q$. Therefore each player plays in exactly one group per round. Consider players (s_1,t_1) and (s_2,t_2). If $s_1 = s_2$, then these players are together in the s_1 group of the $*$ round. If $s_1 \neq s_2$, then these players are together in the γ group of the ρ round, where

$$\begin{aligned} \rho &= (t_1 - t_2) \cdot (s_1 - s_2)^{-1} \\ \gamma &= t_1 - s_1 \cdot \rho \end{aligned}$$

Therefore all the constraints of a schedule are met. □

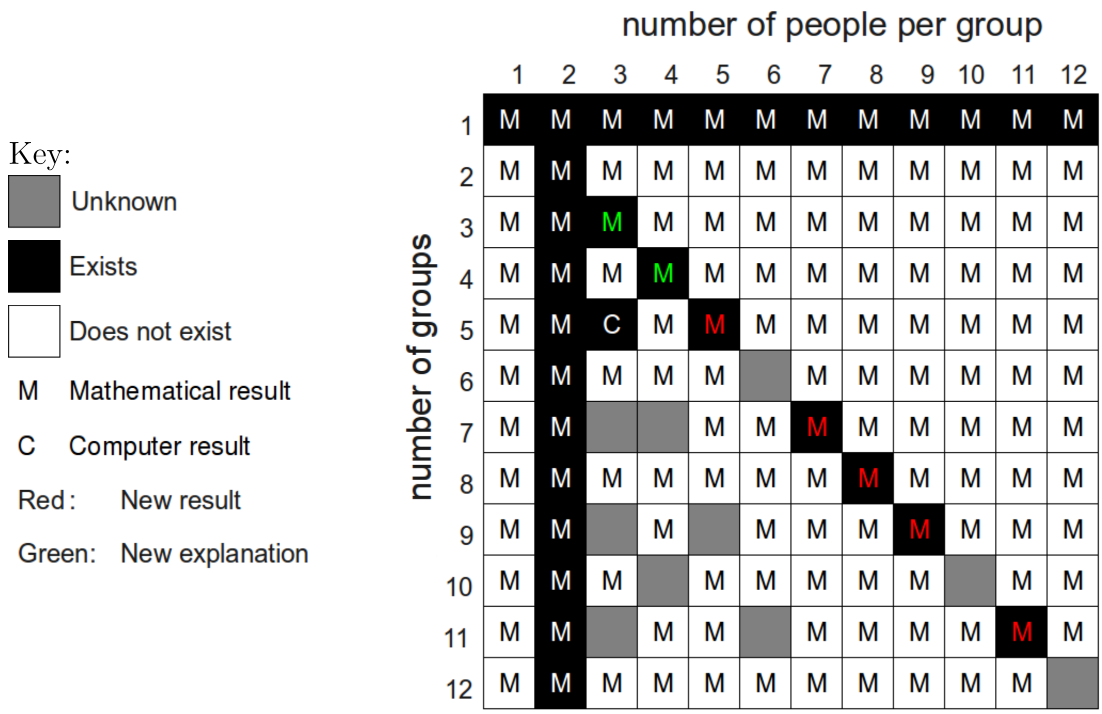


Figure 10: Schedules with $r = 1$, version 5

In Figure 10 we have filled many of the unknown schedules, but a few are still left out. Let's investigate $x = y = 6$, the lowest value on the diagonal that is still unknown.

Theorem 3.4. An $x = y = n$ schedule exists if and only if there exists a set of $n - 1$ mutually orthogonal $n \times n$ Latin squares.[2]

[1]Refer to Appendix A
[2]Refer to Appendix B

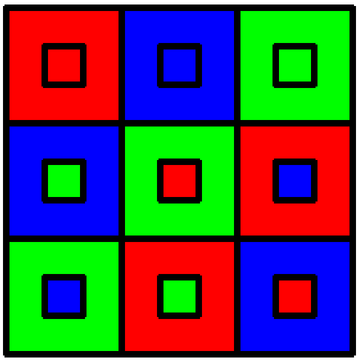
Proof. (⇒) Consider an $x = y = n$ schedule. Choose a group from the first round. Since each of these players have played together, in the following rounds they must be split up among the groups. We can index the groups in each of the rounds after the first round by which player from this group is in it.

Consider any of the remaining $n - 1$ groups from the first round. Similarly, the players in this group are in separate groups in the remaining rounds. Also, each player plays in a group indexed by an element from the first group exactly once, since it must play with this player exactly once. This gives a Latin square where the rounds correspond to rows and the groups correspond to columns. Therefore we have $n - 1$ $n \times n$ Latin squares.

Choose any two of these $n \times n$ Latin squares. Since the players play with each other player exactly once, they must be orthogonal. So we have a set of $n - 1$ mutually orthogonal $n \times n$ Latin squares.

(⇐) Consider a set of $n - 1$ mutually orthogonal $n \times n$ Latin squares. For each $n \times n$ Latin square, the set of elements correspond to a group in the first round. In the following rounds a row corresponds to a round and a column corresponds the the group that it plays in. Therefore we have $n - 1$ groups in the first round and n groups of $n - 1$ players in which everyone plays with everyone else exactly once.

We can add the nth group to the first round. Since these players have already played together, in the following rounds, they need to be separate. Each of the elements in this group stay in the same group in the remaining rounds. They will play with each other player exactly once. Therefore we have an $x = y = n$ schedule in which everyone plays everyone. □



1:	A B C	D E F	G H I
2:	A D G	B E H	C F I
3:	A E I	B F G	C D H
4:	A F H	B D I	C E G

Figure 11: Example of an $x = y = 3$ schedule represented as two 3×3 mutually orthogonal Latin squares

Corollary 3.5. For a prime p and positive integer n, there exists a set of $p^n - 1$ mutually orthogonal $p^n \times p^n$ Latin squares.

Proof. Immediate from Theorems 3.3 and 3.4. □

If an $x = y = 6$ schedule exists, then by Theorem 3.4 there exist 5 mutually orthogonal 6×6 Latin squares. A computer search shows that not even a pair of 6×6 orthogonal Latin squares exist. Therefore an $x = y = 6$ schedule does not exist.

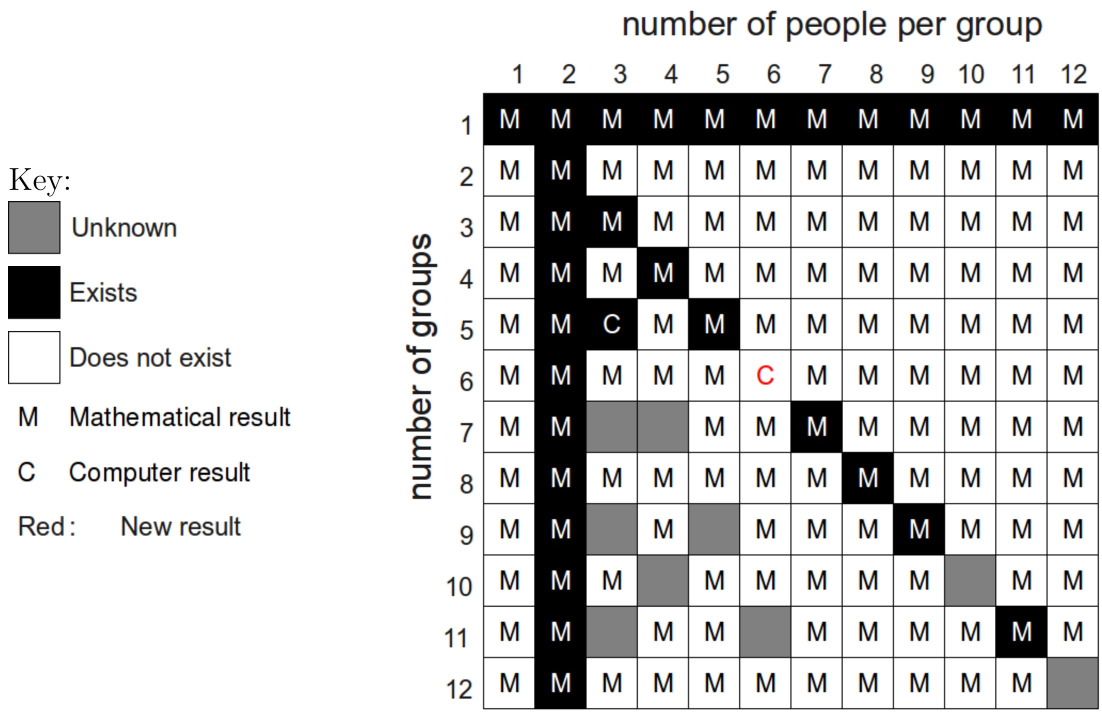


Figure 12: Schedules with $r = 1$, version 6

4 Schedule constructions

In section we show some of the smaller cases where, up to isomorphism, a schedule is unique.

Definition. Two golf schedules are *isomorphic* if there is a way we can manipulate one schedule, by relabeling the players and rearranging the rounds, groups, and players in a group, to get the other schedule.

If we look at two $x = y = 3$ schedules where everyone plays everyone else exactly once

ABC	DEF	GHI	BDG	AEH	FCI
ADG	BEH	CFI	BAC	DEF	GHI
AEI	BFG	CDH	BEI	AFG	DCH
AFH	BDI	CEG	ADI	BFH	CEG

we see that the two schedules with some manipulation are isomorphic.

The problem of representing the same schedule in multiple ways can be reduced. We can arrange the players in each group into alphabetical order, arrange the groups within each round into lexicographical order, and arrange the rounds into lexicographical. This eliminates the problem of rearranging the rounds, groups, or players in a group.

Let's look at a few construction with $r = 1$. Using the labeling power of isomorphism when constructing a schedule we assign the first round in alphabetical order. Let's start with the smallest non-trivial schedule, $x = 2$ and $y = 2$.

Theorem 4.1. The $x = 2$ and $y = 2$ schedule is unique.

Proof. Player A has to to play with C and D in the second and third round. Since, at this point, nothing sets C apart from D, we will have A play with C in the second round and D in the third round. This forces B to play with D in the second round and C in the third round. Therefore we have the following unique schedule in which everyone plays with everyone.

$$\begin{array}{l|l} AB & CD \\ AC & BD \\ AD & BC \end{array}$$

□

Theorem 4.2. The $x = 2$ and $y = 3$ schedule is unique.

Proof. Since A and B play together in the first round, they do not play together again. Let's put A in group 1 and B in group 2 for the remaining rounds. Since A has to play with each other player exactly once we can assign each player to play with A in alphabetical order in the remaining rounds.

$$\begin{array}{l|l|l} AB & CD & EF \\ AC & B & \\ AD & B & \\ AE & B & \\ AF & B & \end{array}$$

If B were to play with D in the second round then E and F would be forced to play together again. That would be a problem. Therefore B needs to play with either E or F, at this point they are the same to B, so we can choose E. Similarly in the third round B will have to play with F.

$$\begin{array}{l|l|l} AB & CD & EF \\ AC & BE & DF \\ AD & BF & CE \\ AE & B & \\ AF & B & \end{array}$$

By the fourth round, B has not played with C or D. If B plays with C, then D and F are forced to play together again. So B must play with D. The rest of the unique schedule follows as shown below.

$$\begin{array}{l|l|l} AB & CD & EF \\ AC & BE & DF \\ AD & BF & CE \\ AE & BD & CF \\ AF & BC & DE \end{array}$$

□

The following are two $x = 2$ and $y = 4$ schedules that are not isomorphic.

AB	CD	EF	GH		AB	CD	EF	GH
AC	BD	EG	FH		AC	BE	GD	HF
AD	BC	EH	FG		AD	BF	GE	HC
AE	BF	CG	DH		AE	BG	CF	DH
AF	BE	CH	DG		AF	BH	CG	DE
AG	BH	CE	DF		AG	BC	EH	FD
AH	BG	CF	DE		AH	BD	EC	FG

Theorem 4.3. The $x = 3$ and $y = 3$ schedule is unique.

Proof. In each of the rounds after the first round A, B, and C must be split up among the three groups.

ABC	DEF	GHI
A	B	C
A	B	C
A	B	C

In the second round, nothing distinguishes elements from the same group in the first round. Therefore, we can choose one player from each group in the first round to go in separate groups in the second round.

ABC	DEF	GHI
ADG	BEH	CFI
A	B	C
A	B	C

Also, player A must play with E and F in separate rounds. Since nothing distinguishes round three from round four we can assign A to play with each in alphabetical order.

ABC	DEF	GHI
ADG	BEH	CFI
AE	B	C
AF	B	C

Player B still needs to play with D and F. If B plays with D in round three, F could not play in either group 2 or 3 in the third round. Therefore B plays with F and the rest follows as shown below. This is a unique construction.

ABC	DEF	GHI
ADG	BEH	CFI
AEI	BFG	CDH
AFH	BDI	CEG

□

5 Future directions

In Section 3 we got a good start on characterizing various values of x and y for $r = 1$, but there are still some unknown values. It would be nice to characterize all values of x and y. Theorem 3.3 proves that the existance of a finite field of order n implies that there exists a schedule such that $x = y = n$. An interesting result would be to show that an $x = y = n$ schedule implies the existance of an order n field.

After we have characterized all values of x and y for $r = 1$, a next step would be to look at various values of x and y for other values or ranges of r. Some concepts that were used in the $r = 1$ case may apply to higher values of r. Obviously if we have a schedule for x and y with r repeats, there exists a schedule for x and y with kr repeats, such that k is a positive integer. This could be done by playing schedules with r repeats k times. Does a schedule exist that doesn't break down into schedules of lower order r?

Given $x = 3$ and $y = 2$ with $r = 2$ repeats, by Theorem 3.1 we have $z = 5$ rounds. In a group in the first round there are $\binom{3}{2} = 3$ pairings that occur for the first time. Since there are 3 players and 2 groups, by the pigeonhole principal, in each of the following rounds one of those pairings must occur. But there are 4 rounds after the first round. At least one of the pairings from this group will occur for the third time by the fifth round. Therefore, a schedule with the given constraints does not exist. This leads to the following question. For a fixed r, does there exist a schedule such that $x > y > 1$?

I started looking for isomorphisms of schedules in Section 4, but there is still a lot left to this question. Continuing this or finding some mathmatical results to characterize isomorphisms of various schedules is another direction.

A Galois fields

Galois fields are used in Theorem 3.3. We begin with a definition of a field and then prove that there exist fields of prime power order.

Definition. A *field* consists of a set, F, and binary operations, addition and multiplication, with the following properties for all $a, b, c, d \in F$ such that $d \neq 0$.

- Closure: $a + b \in F$ and $a \cdot b \in F$
- Associativity: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Commutativity: $a + b = b + a$ and $a \cdot b = b \cdot a$
- Identities: There exist unique elements $0,\ 1 \in F$ such that $0 + a = 1 \cdot a = a$
- Inverses: For each element there exists an oppisite such that $a + (-a) = 0$ and for each nonzero element there exists a reciprical such that $d \cdot d^{-1} = 1$
- Distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$

Examples of fields include $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$ such that p is prime. Notice $\mathbb{Z}$ is not a field since $2 \in \mathbb{Z}$, but $2^{-1} \notin \mathbb{Z}$.

Definition. The *order* of a field F is the number of elements in the set F.

Definition. Let F be a field and $f(x)$ a nonconstant polynomial in $F[x]$. Then $f(x)$ is *irreducible* over F if $f(x)$ cannot be expressed as a product $f(x) = g(x)h(x)$ of polynomials $g(x)$ and $h(x)$ in $F[x]$ both of lower degree than $f(x)$.

Theorem A.1. Given any prime p and any positive integer n, there exists a field F of order p^n

Proof. Let $f(x) \in \mathbb{Z}_p[x]$ be an irreducible polynomial with degree $n+1$. Then $|\mathbb{Z}_p[x]/f(x)| = p^n$, since we have n coefficients each of which can choose from one of p elements. I claim that this set is a field. Clearly it has closure, associativity, commutativity, identities, and distirbutivity under addition and multiplication. Also each element has an additive inverse. We need to show that this has multiplicative inverses.

Consider a polynomial $g(x) \in \mathbb{Z}_p[x]/f(x)$ such that $g(x) \neq 0$. Since $f(x)$ is irreducable, $gcd(g(x), f(x)) = 1$. Then we have polynomials $u(x)$ and $v(x)$ such that $g(x)u(x) + f(x)v(x) = 1$. Since $f(x)v(x) = 0$ we have $g(x)u(x) = 1$ and $u(x)$ is the inverse of $g(x)$. Therefore we have a field of order p^n. □

Definition. Given any prime p and any positive integer n, a field of order p^n is called a *Galois field* of order p^n, denoted $GF(p^n)$. (Note: this field can be shown to be unique)

B Latin squares

Definition. A *Latin square* is an $n \times n$ table such that each cell has an element from a set of n symbols with following constraints:

- Each symbol occurs exactly once in each row.
- Each symbol occurs exactly once in each column.

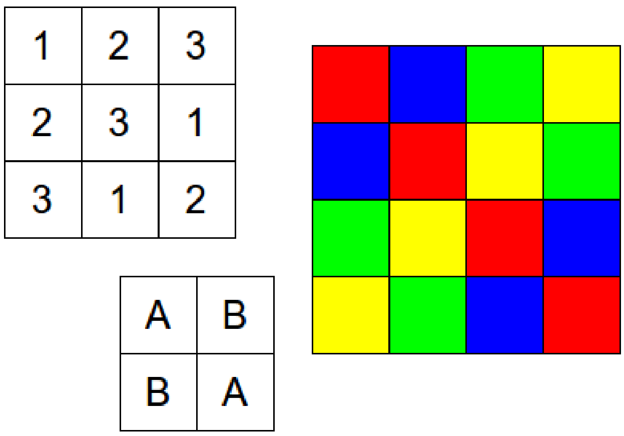


Figure 13: Three examples of Latin squares

Definition. Two Latin squares are *orthogonal* if the pair of symbols from each cell is unique.

Definition. A set of Latin squares are *mutually orthogonal* if every two Latin squares are orthogonal.

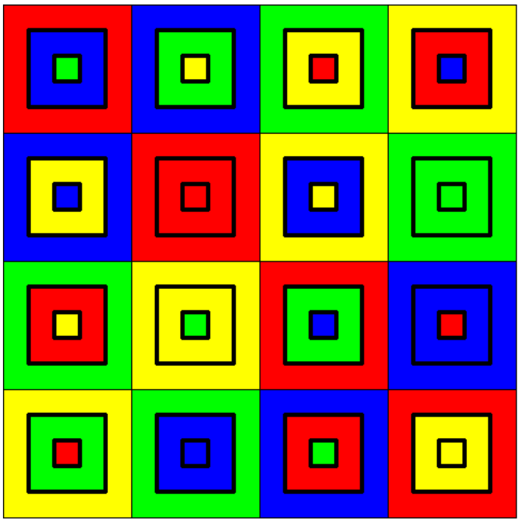


Figure 14: An example of 3 mutually orthogonal 4×4 Latin squares