Golf Scheduling Problems

Tyler Hyndman

Bradley University

ISMAA Annual Meeting
April 3, 2009

Motivating problem

- Setup:
  - 12 players
  - 5 rounds
  - 3 groups of 4

Motivating problem

- Setup:
  - 12 players
  - 5 rounds
  - 3 groups of 4
- Constraint 1:

  Each player plays with each other player at least once

Motivating problem

- Setup:
  - 12 players
  - 5 rounds
  - 3 groups of 4
- Constraint 1:

  Each player plays with each other player at least once
- Constraint 2:

  Each player plays with each other player at most twice

Solution approaches

Solution approaches

- Computer search

Solution approaches

- Computer search
- Combinatorial argument

Computational Complexity

- Let x be the number of players in a group and y be the number of groups

Computational Complexity

- Let x be the number of players in a group and y be the number of groups
- Assign each player an element in a list
- The first x elements are assigned to group 1, the second x group 2, etc.

Computational Complexity

- Let x be the number of players in a group and y be the number of groups
- Assign each player an element in a list
- The first x elements are assigned to group 1, the second x group 2, etc.
- There are $(xy)!$ permutations of that list

Computational Complexity

- Let x be the number of players in a group and y be the number of groups
- Assign each player an element in a list
- The first x elements are assigned to group 1, the second x group 2, etc.
- There are $(xy)!$ permutations of that list
- Divide out $x!$ for each of the y groups

Computational Complexity

- Let x be the number of players in a group and y be the number of groups
- Assign each player an element in a list
- The first x elements are assigned to group 1, the second x group 2, etc.
- There are $(xy)!$ permutations of that list
- Divide out $x!$ for each of the y groups
- Divide out $y!$ for the order of the y groups

Computational Complexity

- Let x be the number of players in a group and y be the number of groups
- Assign each player an element in a list
- The first x elements are assigned to group 1, the second x group 2, etc.
- There are $(xy)!$ permutations of that list
- Divide out $x!$ for each of the y groups
- Divide out $y!$ for the order of the y groups
- Therefore there are $\frac{(xy)!}{(x!)^y y!}$ distinct rounds

Computational Complexity

- Let x be the number of players in a group and y be the number of groups
- Assign each player an element in a list
- The first x elements are assigned to group 1, the second x group 2, etc.
- There are $(xy)!$ permutations of that list
- Divide out $x!$ for each of the y groups
- Divide out $y!$ for the order of the y groups
- Therefore there are $\frac{(xy)!}{(x!)^y y!}$ distinct rounds
- In our specific problem we have $x = 4$ and $y = 3$ which gives us $\frac{12!}{(4!)^3 3!} = 5775$ distinct rounds

Improving the search algorithm

Improvement	Number of computations	Time
None	$5775^5 = 6.42 \times 10^{18}$	203,682 years

Improving the search algorithm

Improvement	Number of computations	Time
None	$5775^5 = 6.42 \times 10^{18}$	203,682 years
Eliminate repeats	$\binom{5775}{5} = 5.34 \times 10^{16}$	1694 years

Improving the search algorithm

Improvement	Number of computations	Time
None	$5775^5 = 6.42 \times 10^{18}$	203,682 years
Eliminate repeats	$\binom{5775}{5} = 5.34 \times 10^{16}$	1694 years
Arbitrary 1st round	$\binom{5775}{4} = 4.63 \times 10^{13}$	1.47 years

Further improvement: 2nd round symmetry

- Do we need to check every single second round?

Further improvement: 2nd round symmetry

- Do we need to check every single second round?
- We can look at symmetrical second rounds

Further improvement: 2nd round symmetry

- Do we need to check every single second round?
- We can look at symmetrical second rounds

Symmetry classes		
$A_0 B_0 C_0 D_0$	$E_0 F_0 G_0 H_0$	$I_0 J_0 K_0 L_0$
$A_0 B_0 C_0 D_0$	$E_1 F_1 G_1 I_3$	$H_3 J_1 K_1 L_1$
$A_0 B_0 C_0 D_0$	$E_2 F_2 I_2 J_2$	$G_2 H_2 K_2 L_2$
$A_1 B_1 C_1 E_3$	$D_3 I_1 J_1 K_1$	$F_1 G_1 H_1 L_3$
$A_1 B_1 C_1 E_3$	$D_3 F_2 G_2 I_3$	$H_3 J_1 K_1 L_1$
$A_1 B_1 C_1 E_3$	$D_3 F_3 I_2 J_2$	$G_2 H_2 K_2 L_2$
$A_2 B_2 E_2 F_2$	$C_2 D_2 I_2 J_2$	$G_2 H_2 K_2 L_2$
$A_2 B_2 E_2 F_2$	$C_3 G_3 I_2 J_2$	$D_3 H_3 K_2 L_2$
$A_2 B_2 E_3 I_3$	$C_3 F_2 G_2 J_3$	$D_3 H_3 K_2 L_2$

Improving the search algorithm

Improvement	Number of computations	Time
None	$5775^5 = 6.42 \times 10^{18}$	203,682 years
Eliminate repeats	$\binom{5775}{5} = 5.34 \times 10^{16}$	1694 years
Arbitrary 1st round	$\binom{5775}{4} = 4.63 \times 10^{13}$	1.47 years
2nd round symmetries	$8 \cdot \binom{5775}{3} = 3.21 \times 10^{10}$	3 days

Further improvement: check if still possible

- After choose two, three, or four rounds we can check if both constraints can still be met

Further improvement: check if still possible

- After choose two, three, or four rounds we can check if both constraints can still be met
- Constraint 1:
  - There are $\binom{4}{2} \cdot 3 = 18$ pairs in a round
  - If more pairings are required than allowed in the remaining rounds, do not continue

Further improvement: check if still possible

- After choose two, three, or four rounds we can check if both constraints can still be met
- Constraint 1:
  - There are $\binom{4}{2} \cdot 3 = 18$ pairs in a round
  - If more pairings are required than allowed in the remaining rounds, do not continue
- Constraint 2:
  - Check the number of times each pairings has occured
  - If a pairing has occured three or more times, do not continue

Improving the search algorithm

Improvement	Number of computations	Time
None	$5775^5 = 6.42 \times 10^{18}$	203,682 years
Eliminate repeats	$\binom{5775}{5} = 5.34 \times 10^{16}$	1694 years
Arbitrary 1st round	$\binom{5775}{4} = 4.63 \times 10^{13}$	1.47 years
2nd round symmetries	$9 \cdot \binom{5775}{3} = 3.21 \times 10^{10}$	3 days

Is it possible for . . .

	Pairings needed?	Pairings possible?
one person to play all?		
everyone to play everyone?		

Is it possible for . . .

	Pairings needed?	Pairings possible?
one person to play all?	11	
everyone to play everyone?		

Is it possible for . . .

	Pairings needed?	Pairings possible?
one person to play all?	11	$3 \cdot 5 = 15$
everyone to play everyone?		

Is it possible for . . .

	Pairings needed?	Pairings possible?
one person to play all?	11	$3 \cdot 5 = 15$
everyone to play everyone?	$\binom{12}{2} = 66$	

Is it possible for . . .

	Pairings needed?	Pairings possible?
one person to play all?	11	$3 \cdot 5 = 15$
everyone to play everyone?	$\binom{12}{2} = 66$	$\binom{4}{2} \cdot 3 \cdot 5 = 90$

Combinatorial argument

- 90 pairings in total
- 66 pairings needed for everyone to play everyone once
- Then 24 of the 90 are the second occurrence of a pairing

Combinatorial argument

- 90 pairings in total
- 66 pairings needed for everyone to play everyone once
- Then 24 of the 90 are the second occurrence of a pairing
- If we split a group from one round into the groups in the following rounds, by the pigeonhole principle, at least one of the pairings will repeat in each round

Combinatorial argument

- 90 pairings in total
- 66 pairings needed for everyone to play everyone once
- Then 24 of the 90 are the second occurrence of a pairing
- If we split a group from one round into the groups in the following rounds, by the pigeonhole principle, at least one of the pairings will repeat in each round
- Then we have at least $3(4+3+2+1) = 30$ forced second occurrences of pairings

Combinatorial argument

- 90 pairings in total
- 66 pairings needed for everyone to play everyone once
- Then 24 of the 90 are the second occurrence of a pairing
- If we split a group from one round into the groups in the following rounds, by the pigeonhole principle, at least one of the pairings will repeat in each round
- Then we have at least $3(4 + 3 + 2 + 1) = 30$ forced second occurrences of pairings
- We can relax Constraint 2 to allow pairings to occur at most three times

Solution with Constraint 1 and relaxed Constraint 2 Round 1

ABCD | *EFGH* | *IJKL*

Relations												
A:		*B*	*C*	*D*								
B:	*A*		*C*	*D*								
C:	*A*	*B*		*D*								
D:	*A*	*B*	*C*									
E:						*F*	*G*	*H*				
F:					*E*		*G*	*H*				
G:					*E*	*F*		*H*				
H:					*E*	*F*	*G*					
I:										*J*	*K*	*L*
J:									*I*		*K*	*L*
K:									*I*	*J*		*L*
L:									*I*	*J*	*K*	

Solution with Constraint 1 and relaxed Constraint 2 Round 2

$ABCD \mid EFGH \mid IJKL$

$ABEI \mid CFGJ \mid DHKL$

Relations												
A:		B	C	D	E				I			
B:	A		C	D	E				I			
C:	A	B		D		F	G			J		
D:	A	B	C					H			K	L
E:	A	B				F	G	H	I			
F:			C		E		G	H		J		
G:			C		E	F		H		J		
H:				D	E	F	G				K	L
I:	A	B			E					J	K	L
J:			C			F	G		I		K	L
K:				D				H	I	J		L
L:				D				H	I	J	K	

Solution with Constraint 1 and relaxed Constraint 2 Round 3

ABCD	*EFGH*	*IJKL*
ABEI	*CFGJ*	*DHKL*
ABFK	*CEHL*	*DGIJ*

Relations												
A:		*B*	*C*	*D*	*E*	*F*			*I*		*K*	
B:	*A*		*C*	*D*	*E*	*F*			*I*		*K*	
C:	*A*	*B*		*D*	*E*	*F*	*G*	*H*		*J*		*L*
D:	*A*	*B*	*C*				*G*	*H*	*I*	*J*	*K*	*L*
E:	*A*	*B*	*C*			*F*	*G*	*H*	*I*			*L*
F:	*A*	*B*	*C*		*E*		*G*	*H*		*J*	*K*	
G:			*C*	*D*	*E*	*F*		*H*	*I*	*J*		
H:			*C*	*D*	*E*	*F*	*G*				*K*	*L*
I:	*A*	*B*		*D*	*E*		*G*			*J*	*K*	*L*
J:			*C*	*D*		*F*	*G*		*I*		*K*	*L*
K:	*A*	*B*		*D*		*F*		*H*	*I*	*J*		*L*
L:			*C*	*D*	*E*			*H*	*I*	*J*	*K*	

Solution with Constraint 1 and relaxed Constraint 2
Round 4

ABCD | *EFGH* | *IJKL*
ABEI | *CFGJ* | *DHKL*
ABFK | *CEHL* | *DGIJ*
AEHJ | *BGKL* | *CDFI*

Relations												
A:		B	C	D	E	F		H	I	J	K	
B:	A		C	D	E	F	G		I		K	L
C:	A	B		D	E	F	G	H	I	J		L
D:	A	B	C			F	G	H	I	J	K	L
E:	A	B	C			F	G	H	I	J		L
F:	A	B	C	D	E		G	H	I	J	K	
G:		B	C	D	E	F		H	I	J	K	L
H:	A		C	D	E	F	G			J	K	L
I:	A	B	C	D	E	F	G			J	K	L
J:	A		C	D	E	F	G	H	I		K	L
K:	A	B		D		F	G	H	I	J		L
L:		B	C	D	E		G	H	I	J	K	

Solution with Constraint 1 and relaxed Constraint 2
Round 5

ABCD	*EFGH*	*IJKL*
ABEI	*CFGJ*	*DHKL*
ABFK	*CEHL*	*DGIJ*
AEHJ	*BGKL*	*CDFI*
AFGL	*BHIJ*	*CDEK*

Relations												
A:		*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
B:	*A*		*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
C:	*A*	*B*		*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
D:	*A*	*B*	*C*		*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
E:	*A*	*B*	*C*	*D*		*F*	*G*	*H*	*I*	*J*	*K*	*L*
F:	*A*	*B*	*C*	*D*	*E*		*G*	*H*	*I*	*J*	*K*	*L*
G:	*A*	*B*	*C*	*D*	*E*	*F*		*H*	*I*	*J*	*K*	*L*
H:	*A*	*B*	*C*	*D*	*E*	*F*	*G*		*I*	*J*	*K*	*L*
I:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*		*J*	*K*	*L*
J:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*		*K*	*L*
K:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*		*L*
L:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	

Solution with Constraint 1 and relaxed Constraint 2 Number of repeated pairings

ABCD	*EFGH*	*IJKL*
ABEI	*CFGJ*	*DHKL*
ABFK	*CEHL*	*DGIJ*
AEHJ	*BGKL*	*CDFI*
AFGL	*BHIJ*	*CDEK*

- Black - one pairing
- Green - two pairings
- Blue - three pairings

Relations												
A:		*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
B:	*A*		*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
C:	*A*	*B*		*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
D:	*A*	*B*	*C*		*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
E:	*A*	*B*	*C*	*D*		*F*	*G*	*H*	*I*	*J*	*K*	*L*
F:	*A*	*B*	*C*	*D*	*E*		*G*	*H*	*I*	*J*	*K*	*L*
G:	*A*	*B*	*C*	*D*	*E*	*F*		*H*	*I*	*J*	*K*	*L*
H:	*A*	*B*	*C*	*D*	*E*	*F*	*G*		*I*	*J*	*K*	*L*
I:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*		*J*	*K*	*L*
J:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*		*K*	*L*
K:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*		*L*
L:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	

Counting rounds

- A player has to play with $xy - 1$ other players

Counting rounds

- A player has to play with $xy - 1$ other players
- In each round they can play with $x - 1$ other players

Counting rounds

- A player has to play with $xy - 1$ other players
- In each round they can play with $x - 1$ other players
- Let r be the number of times each pairing occurs

Counting rounds

- A player has to play with $xy - 1$ other players
- In each round they can play with $x - 1$ other players
- Let r be the number of times each pairing occurs
- Then $r\frac{xy-1}{x-1}$ rounds are needed

Single pairings

- Let us now investigate when each pair of players occurs exactly once

Single pairings

- Let us now investigate when each pair of players occurs exactly once
- Recall $\frac{xy-1}{x-1}$ is the number of round required
- This number must be an integer

Single pairings

- Let us now investigate when each pair of players occurs exactly once
- Recall $\frac{xy-1}{x-1}$ is the number of round required
- This number must be an integer
- Special cases:
  - If $x = 2$ we have $2y - 1$ rounds
  - If $x = y$ we have $\frac{x^2-1}{x-1} = x + 1$ rounds
  - if $x = 3$ and $y = 5$, $\frac{(3)(5)-1}{3-1} = 7$ (Kirkman's School Girl Problem)

Known constructions with single pairings

- Kirkman's School Girl Problem

Known constructions with single pairings

- Kirkman's School Girl Problem
- $x = 2$ and arbitrary y

Known constructions with single pairings

- Kirkman's School Girl Problem
- $x = 2$ and arbitrary y
- $x = y = p$ such that p is prime

Known constructions with single pairings

- Kirkman's School Girl Problem
- $x = 2$ and arbitrary y
- $x = y = p$ such that p is prime
- $x = y = 4$ in 5 rounds

Future directions

- Try to construct a schedule for all $x = y$ single pairings in $x + 1$ rounds

Future directions

- Try to construct a schedule for all $x = y$ single pairings in $x + 1$ rounds
- The necessary condition, $r \cdot \frac{xy-1}{x-1} \in \mathbb{Z}$, for everyone to play everyone else exactly r
  - Determine whether it is sufficient
  - If not, find additional conditions to characterize