

Design Theory and Algorithms

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Golf Problem

- ▶ Setup:
 - ▶ 12 players
 - ▶ 5 rounds
 - ▶ 3 groups of 4

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 - ▶ 5 rounds
 - ▶ 3 groups of 4
- ▶ Constraint 1:
 - Each player plays with each other player at least once

Golf Problem

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 - ▶ 12 players
 - ▶ 5 rounds
 - ▶ 3 groups of 4
- ▶ Constraint 1:
Each player plays with each other player at least once
- ▶ Constraint 2:
Each player plays with each other player at most twice

How many distinct rounds?

- ▶ Assign each person a letter (A-L)
- ▶ The letters can be arranged in a list
- ▶ The first four letters are in group 1, second four in group 2, and third 4 in group 3

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- ▶ There are $12!$ permutations of that list
- ▶ Divide out $4!$ for each group
- ▶ Divide out $3!$ for the order of the three groups
- ▶ Therefore there are $\frac{12!}{4!4!4!3!} = 5775$ different rounds

Simple Program 1

```
allrounds := MakeArrayOfRounds
for every v from 1 to 5775
  set round[1] to allrounds[v]
  for every w from 1 to 5775
    set round[2] to allrounds[w]
    for every x from 1 to 5775
      set round[3] to allrounds[x]
      for every y from 1 to 5775
        set round[4] allrounds[y]
        for every z from 1 to 5775
          set round[5] to allrounds[z]
          if both constraints are satisfied
            Success!
```

Improve program

- ▶ Stop counting repeats
- ▶ Assign the first round $ABCD \mid EFGH \mid IJKL$

Simple Program 2

```
allrounds := MakeArrayOfRounds
round[1] = allrounds[0]
for every w from 1 to 5775
    set round[2] to allrounds[w]
    for every x from w+1 to 5775
        set round[3] to allrounds[x]
        for every y from x+1 to 5775)
            set round[4] to allrounds[y]
            for every z from y+1 to 5775
                set round[5] to allrounds[z];
                if both constraints are satisfied
                    Success!
```

Computational time of brute force

| Number of computations | 1 microsecond per computation |
|---|-------------------------------|
| $5775^5 = 6.42 \times 10^{18}$ | 203,682 years |
| $\binom{5775}{5} = 5.34 \times 10^{16}$ | 1694 years |
| $\binom{5775}{4} = 4.63 \times 10^{13}$ | 1.47 years |
| $\binom{5775}{3} = 3.21 \times 10^{10}$ | 8.9 hours |
| $\binom{5775}{2} = 1.67 \times 10^7$ | .28 mins |

Is it possible for . . .

| | Pairings needed? | Pairings possible? |
|----------------------------|------------------|--------------------|
| one person to play all? | | |
| everyone to play everyone? | | |

Is it possible for . . .

| | Pairings needed? | Pairings possible? |
|----------------------------|------------------|--------------------|
| one person to play all? | 11 | |
| everyone to play everyone? | | |

Is it possible for . . .

| | Pairings needed? | Pairings possible? |
|----------------------------|------------------|--------------------|
| one person to play all? | 11 | 15 |
| everyone to play everyone? | | |

Is it possible for . . .

| | Pairings needed? | Pairings possible? |
|----------------------------|------------------|--------------------|
| one person to play all? | 11 | 15 |
| everyone to play everyone? | 66 | |

Is it possible for . . .

| | Pairings needed? | Pairings possible? |
|----------------------------|------------------|--------------------|
| one person to play all? | 11 | 15 |
| everyone to play everyone? | 66 | 90 |

New relations in a group for second round

- ▶ Always use first round: $ABCD \mid EFGH \mid IJKL$
- ▶ In the second round
 - ▶ Group $ABCD$ has no new relations
 - ▶ Group $ABCE$ has 6 new relations
 - ▶ Group $ABEF$ has 8 new relations
 - ▶ Group $ABEI$ has 10 new relations

Symmetries of round 2

| New Relations | Example | | |
|---------------|-------------------|-------------------|-------------------|
| 0 | $A_0 B_0 C_0 D_0$ | $E_0 F_0 G_0 H_0$ | $I_0 J_0 K_0 L_0$ |
| 12 | $A_0 B_0 C_0 D_0$ | $E_1 F_1 G_1 I_3$ | $H_3 J_1 K_1 L_1$ |
| 16 | $A_0 B_0 C_0 D_0$ | $E_2 F_2 I_2 J_2$ | $G_2 H_2 K_2 L_2$ |
| 18 | $A_1 B_1 C_1 E_3$ | $D_3 I_1 J_1 K_1$ | $F_1 G_1 H_1 L_3$ |
| 22 | $A_1 B_1 C_1 E_3$ | $D_3 F_2 G_2 I_3$ | $H_3 J_1 K_1 L_1$ |
| 24 | $A_1 B_1 C_1 E_3$ | $D_3 F_3 I_2 J_2$ | $G_2 H_2 K_2 L_2$ |
| 24 | $A_2 B_2 E_2 F_2$ | $C_2 D_2 I_2 J_2$ | $G_2 H_2 K_2 L_2$ |
| 28 | $A_2 B_2 E_2 F_2$ | $C_3 G_3 I_2 J_2$ | $D_3 H_3 K_2 L_2$ |
| 30 | $A_2 B_2 E_3 I_3$ | $C_3 F_2 G_2 J_3$ | $D_3 H_3 K_2 L_2$ |

Simple Program 3

```
allrounds := MakeArrayOfRounds, set round[1] to allrounds[1]
for every v in symmetry set
  set round[2] to allrounds[v]
  for every w from 1 to 5775
    set round[2] to allrounds[w]
    if both constraints still possible  $\Rightarrow$  keep searching
    for every x from w+1 to 5775
      set round[3] to allrounds[x]
      if both constraints still possible  $\Rightarrow$  keep searching
      for(y from x+1 to 5775)
        set round[4] to allrounds[y]
        if both constraints still possible  $\Rightarrow$  keep searching
        for every z from y+1 to 5775
          set round[5] to allrounds[z]
          if both constraints are satisfied  $\Rightarrow$  Success!!
```

Solution with Constraint 1 - Round 1

ABCD | EFGH | IJKL

| Relations | | | | | | | | | | | |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| A: | | <i>B</i> | <i>C</i> | <i>D</i> | | | | | | | |
| B: | <i>A</i> | | <i>C</i> | <i>D</i> | | | | | | | |
| C: | <i>A</i> | <i>B</i> | | <i>D</i> | | | | | | | |
| D: | <i>A</i> | <i>B</i> | <i>C</i> | | | | | | | | |
| E: | | | | | <i>F</i> | <i>G</i> | <i>H</i> | | | | |
| F: | | | | | <i>E</i> | <i>G</i> | <i>H</i> | | | | |
| G: | | | | | <i>E</i> | <i>F</i> | <i>H</i> | | | | |
| H: | | | | | <i>E</i> | <i>F</i> | <i>G</i> | | | | |
| I: | | | | | | | | <i>J</i> | <i>K</i> | <i>L</i> | |
| J: | | | | | | | | <i>I</i> | <i>K</i> | <i>L</i> | |
| K: | | | | | | | | <i>I</i> | <i>J</i> | | <i>L</i> |
| L: | | | | | | | | <i>I</i> | <i>J</i> | <i>K</i> | |

Solution with Constraint 1 - Round 2

ABCD | *EFGH* | *IJKL*
ABEF | *CDIJ* | *GHLK*

| Relations | | | | | | | | | | | |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| A: | | <i>B</i> | <i>C</i> | <i>D</i> | <i>E</i> | <i>F</i> | | | | | |
| B: | <i>A</i> | | <i>C</i> | <i>D</i> | <i>E</i> | <i>F</i> | | | | | |
| C: | <i>A</i> | <i>B</i> | | <i>D</i> | | | | <i>I</i> | <i>J</i> | | |
| D: | <i>A</i> | <i>B</i> | <i>C</i> | | | | | <i>I</i> | <i>J</i> | | |
| E: | <i>A</i> | <i>B</i> | | | | <i>F</i> | <i>G</i> | <i>H</i> | | | |
| F: | <i>A</i> | <i>B</i> | | | <i>E</i> | | <i>G</i> | <i>H</i> | | | |
| G: | | | | | <i>E</i> | <i>F</i> | | <i>H</i> | | <i>K</i> | <i>L</i> |
| H: | | | | | <i>E</i> | <i>F</i> | <i>G</i> | | | <i>K</i> | <i>L</i> |
| I: | | | <i>C</i> | <i>D</i> | | | | | <i>J</i> | <i>K</i> | <i>L</i> |
| J: | | | <i>C</i> | <i>D</i> | | | | <i>I</i> | | <i>K</i> | <i>L</i> |
| K: | | | | | | | <i>G</i> | <i>H</i> | <i>I</i> | <i>J</i> | <i>L</i> |
| L: | | | | | | | <i>G</i> | <i>H</i> | <i>I</i> | <i>J</i> | <i>K</i> |

Solution with Constraint 1 - Round 3

ABCD | *EFGH* | *IJKL*
ABEF | *CDIJ* | *GHLK*
ABGI | *CEHJ* | *DFKL*

| Relations | | | | | | | | | | | | |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| A: | | <i>B</i> | <i>C</i> | <i>D</i> | <i>E</i> | <i>F</i> | <i>G</i> | | <i>I</i> | | | |
| B: | <i>A</i> | | <i>C</i> | <i>D</i> | <i>E</i> | <i>F</i> | <i>G</i> | | <i>I</i> | | | |
| C: | <i>A</i> | <i>B</i> | | <i>D</i> | <i>E</i> | | | <i>H</i> | <i>I</i> | <i>J</i> | | |
| D: | <i>A</i> | <i>B</i> | <i>C</i> | | | <i>F</i> | | | <i>I</i> | <i>J</i> | <i>K</i> | <i>L</i> |
| E: | <i>A</i> | <i>B</i> | <i>C</i> | | | <i>F</i> | <i>G</i> | <i>H</i> | | <i>J</i> | | |
| F: | <i>A</i> | <i>B</i> | | <i>D</i> | <i>E</i> | | <i>G</i> | <i>H</i> | | | <i>K</i> | <i>L</i> |
| G: | <i>A</i> | <i>B</i> | | | <i>E</i> | <i>F</i> | | <i>H</i> | <i>I</i> | | <i>K</i> | <i>L</i> |
| H: | | | <i>C</i> | | <i>E</i> | <i>F</i> | <i>G</i> | | | <i>J</i> | <i>K</i> | <i>L</i> |
| I: | <i>A</i> | <i>B</i> | <i>C</i> | <i>D</i> | | | <i>G</i> | | | <i>J</i> | <i>K</i> | <i>L</i> |
| J: | | | <i>C</i> | <i>D</i> | <i>E</i> | | | <i>H</i> | <i>I</i> | | <i>K</i> | <i>L</i> |
| K: | | | | <i>D</i> | | <i>F</i> | <i>G</i> | <i>H</i> | <i>I</i> | <i>J</i> | | <i>L</i> |
| L: | | | | <i>D</i> | | <i>F</i> | <i>G</i> | <i>H</i> | <i>I</i> | <i>J</i> | <i>K</i> | |

Solution with Constraint 1 - Round 4

ABCD | *EFGH* | *IJKL*
ABEF | *CDIJ* | *GHKL*
ABGI | *CEHJ* | *DFKL*
ABKL | *CFGJ* | *DEHI*

| Relations | | | | | | | | | | | | |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| A: | | <i>B</i> | C | D | E | F | G | | I | | <i>K</i> | <i>L</i> |
| B: | <i>A</i> | | C | D | E | F | G | | I | | <i>K</i> | <i>L</i> |
| C: | A | B | | D | E | <i>F</i> | <i>G</i> | H | I | <i>J</i> | | |
| D: | A | B | C | | <i>E</i> | F | | <i>H</i> | <i>I</i> | J | K | L |
| E: | A | B | C | <i>D</i> | | F | G | <i>H</i> | <i>I</i> | J | | |
| F: | A | B | <i>C</i> | D | E | | <i>G</i> | H | | <i>J</i> | K | L |
| G: | A | B | <i>C</i> | | E | <i>F</i> | | H | I | <i>J</i> | K | L |
| H: | | | C | <i>D</i> | <i>E</i> | F | G | | <i>I</i> | J | K | L |
| I: | A | B | C | <i>D</i> | <i>E</i> | | G | <i>H</i> | | J | K | L |
| J: | | | <i>C</i> | D | E | <i>F</i> | <i>G</i> | H | I | | K | L |
| K: | <i>A</i> | <i>B</i> | | D | | F | G | H | I | J | | <i>L</i> |
| L: | <i>A</i> | <i>B</i> | | D | | F | G | H | I | J | <i>K</i> | |

Solution with Constraint 1 - Round 5

ABCD | *EFGH* | *IJKL*
ABEF | *CDIJ* | *GHKL*
ABGI | *CEHJ* | *DFKL*
ABKL | *CFGJ* | *DEHI*
ABHJ | *CEKL* | *DFGI*

| Relations | | | | | | | | | | | | |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| A: | | <i>B</i> | C | D | E | F | G | <i>H</i> | I | <i>J</i> | K | L |
| B: | <i>A</i> | | C | D | E | F | G | <i>H</i> | I | <i>J</i> | K | L |
| C: | A | B | | D | <i>E</i> | F | G | H | I | J | <i>K</i> | <i>L</i> |
| D: | A | B | C | | E | <i>F</i> | <i>G</i> | H | <i>I</i> | J | K | L |
| E: | A | B | <i>C</i> | D | | F | G | H | I | J | <i>K</i> | <i>L</i> |
| F: | A | B | C | <i>D</i> | E | | <i>G</i> | H | <i>I</i> | J | K | L |
| G: | A | B | C | <i>D</i> | E | <i>F</i> | | H | <i>I</i> | J | K | L |
| H: | <i>A</i> | <i>B</i> | C | D | E | F | G | | I | <i>J</i> | K | L |
| I: | A | B | C | <i>D</i> | E | <i>F</i> | <i>G</i> | H | | J | K | L |
| J: | <i>A</i> | <i>B</i> | C | D | E | F | G | <i>H</i> | I | | K | L |
| K: | A | B | <i>C</i> | D | <i>E</i> | F | G | H | I | J | | <i>L</i> |
| L: | A | B | <i>C</i> | D | <i>E</i> | F | G | H | I | J | <i>K</i> | |

Future Questions

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- ▶ Find a counting argument.

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- ▶ Find a counting argument.
- ▶ Generalize: x groups of y people play z rounds.